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LETTER TO THE EDITOR

SO (1, 3) symmetry and Coulomb scattering

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Abstract. The Coulomb S-matrix is calculated via spherical functions on a hyperboloid. Relations with the standard results are discussed, and further prospects of this viewpoint are noted.

It is well known that the unbounded states of the non-relativistic Coulomb problem possess an SO(1, 3) symmetry. Bander and Itzykson (1966) have given a construction relating the eigenfunctions of the Schrödinger equation with the spherical functions on a hyperboloid, on which the SO(1, 3) group acts naturally. Here we will calculate the S-matrix elements from the spherical functions and the result is, modulo a factor depending only on the total energy, the same as what is given in any standard quantum mechanics text book (cf Landau and Lifshitz 1977). As pointed out by Hörmander (1976) and others, in calculating the S-matrix from the time-dependent scattering theory for long-range potentials, the phase shift is determined only up to a factor which may depend on the total energy as above. Our result, although not 'standard', is in agreement with the relativistic version: Herbst (1974) has written down explicitly the scattering operator for the spinless relativistic Coulomb problem up to first order in $\alpha = \frac{1}{137}$. By taking the non-relativistic limit, he gets an operator which differs from the non-relativistic Coulomb scattering operator by a factor of $\Gamma(1+\mathrm{i}p)/\Gamma(1-\mathrm{i}p)$, where p is the reciprocal energy. From our calculation below, we have indeed

$$S_l = \frac{\Gamma(l+1-\mathrm{i}p)}{\Gamma(l+1+\mathrm{i}p)} \frac{\Gamma(1+\mathrm{i}p)}{\Gamma(1-\mathrm{i}p)} \tag{1}$$

the first term being the 'standard' result. Let

$$H = \{y_0, y_1, y_2, y_3 \in R^4 \,|\, y_0^2 - y_1^2 - y_2^2 - y_3^2 = 1, y_0 > 0\}$$

denote the one sheeted unit hyperboloid. It is convenient to write $y \in H$ as $(y_0, (y_0^2-1)^{1/2}\Omega)$ where $\Omega \in S^2 \subset R^3$ is a unit vector. The spherical functions on H are of the form

$$(y_0^2-1)^{-1/4} B_{\mathrm{i}p-1/2}^{-l-1/2}(y_0)\, Y_{lm}(\Omega) \tag{2}$$

where B is the associated Legendre function of the first kind, and Y is the spherical harmonics on S^2. Define

$$\begin{aligned}[y, \theta] &= y_0 - y_1\theta_1 - y_2\theta_2 - y_3\theta_3 \\ &= y_0 - (y_0^2-1)^{1/2}\langle \Omega, \theta \rangle. \end{aligned} \tag{3}$$

Here we can either view $\theta=(\theta_1, \theta_2, \theta_3)\in S^2$, a unit vector in R^3, or $\theta=(1, \theta_1, \theta_2, \theta_3)$, $[,]$ the Lorentz inner product $(+---)$, and $[\theta, \theta]=0$.

From Vilenkin (1968, p 529) we have the following identity

$$(\cosh u-\sinh u\cos v)^{\mathrm{i}p-1}=2(2\pi)^{1/2}\Gamma(\mathrm{i}p)$$
$$\times(\sinh u)^{-1/2}\sum_{l=0}^{\infty}\frac{(-1)^l(2l+1)}{\Gamma(\mathrm{i}p-l)}B_{\mathrm{i}p-1/2}^{-l-1/2}(\cosh u)P_l(\cos v) \qquad (4)$$

substituting

$$y_0=\cosh u, \qquad (y_0^2-1)^{1/2}=\sinh u, \qquad \langle\Omega, \theta\rangle=\cos v \qquad (5)$$

into (4), and using

$$P_l(\langle\Omega, \theta\rangle)=(4\pi)^{-1}\sum Y_{lm}(\Omega)Y_{lm}^*(\theta) \qquad (6)$$

we obtain

$$[y, \theta]^{\mathrm{i}p-1}=\frac{1}{(2\pi)^{1/2}}\Gamma(\mathrm{i}p)(y_0^2-1)^{-1/4}\sum\frac{(-1)^l}{\Gamma(\mathrm{i}p-l)}B_{\mathrm{i}p-1/2}^{-l-1/2}(y_0)Y_{lm}(\Omega)Y_{lm}^*(\theta) \qquad (7)$$

which is a linear combination of the spherical functions (2).

The function $[y, \theta]^{\mathrm{i}p-1}$ plays the same role on H as $\exp(\mathrm{i}\langle x, k\rangle)$ on R^3: these are eigenfunctions for the Laplace-Beltrami operator on H; they form irreducible subspaces for the SO(1, 3) representation for fixed real p; they are the basis for the Fourier expansion on H (for details see Gelfand *et al* 1966). In light of the above, it is natural to consider the incoming states as

$$\psi_k^+=[y, \theta]^{\mathrm{i}p-1} \qquad (8)$$

where

$$|k|=p^{-1}, \qquad k=|k|\theta. \qquad (9)$$

Using

$$\psi^-=(\psi^+)^* \qquad (10)$$

we have

$$\psi_k^-=[y, \theta]^{-\mathrm{i}p-1} \qquad (11)$$

and

$$S(k, k')=\langle\psi_k^+, \psi_{k'}^-\rangle$$
$$=\int_H[y, \theta]^{\mathrm{i}p-1}[y, \theta']^{\mathrm{i}p'-1}\,\mathrm{d}y \qquad (12)$$

where $\mathrm{d}y$ is the SO(1, 3) invariant volume form on H

$$\mathrm{d}y=(y_0^2-1)^{1/2}\,\mathrm{d}y_0\mathrm{d}\Omega. \qquad (13)$$

We apply Mehler's inversion formula (Magnus *et al* 1966) and some identities for the Γ function and obtain

$$S(k, k')=\frac{1}{2\pi}\frac{\delta_p(p')}{pp'}\sum\frac{\Gamma(l+1-\mathrm{i}p)}{\Gamma(l+1+\mathrm{i}p)}\frac{\Gamma(1+\mathrm{i}p)}{\Gamma(1-\mathrm{i}p)}Y_{lm}(\theta)Y_{ln}^*(\theta'). \qquad (14)$$

Thus

$$S_l = \frac{\Gamma(l+1-\mathrm{i}p)}{\Gamma(l+1+\mathrm{p})} \frac{\Gamma(1+\mathrm{i}p)}{\Gamma(1-\mathrm{i}p)} \tag{15}$$

as claimed in (1).

In drawing the analogy between $[y, \theta]^{\mathrm{i}p-1}$ and $\exp(\mathrm{i}\langle x, k\rangle)$, we can also interpret these as free plane waves, corresponding classically to a family of parallel geodesics on H. This point of view will be helpful when one considers the problem of perturbation of Coulomb potential or scattering for Coulomb plus a short range potential. Here we are using the fact that the geodesic flow on H is canonical related to the Hamiltonian flow of the positive energy Coulomb potential.

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